NORMALIZERS OF PARABOLIC SUBGROUPS OF COXETER GROUPS

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Abstract. We improve a bound of Borcherds on the virtual cohomological dimension of the non-reflection part of the normalizer of a parabolic subgroup of a Coxeter group. Our bound is in terms of the types of the components of the corresponding Coxeter subdiagram rather than the number of nodes. A consequence is an extension of Brink's result that the non-reflection part of a reflection centralizer is free. Namely, the non-reflection part of the normalizer of parabolic subgroup of type D_5 or $A_{m\,\mathrm{odd}}$ is either free or has a free subgroup of index 2.

Suppose Π is a Coxeter diagram, J is a subdiagram and $W_J \subseteq W_\Pi$ is the corresponding inclusion of Coxeter groups. The normalizer $N_{W_\Pi}(W_J)$ has been described in detail by Borcherds [3] and Brink-Howlett [5]. Such normalizers have significant applications to working out the automorphism groups of Lorentzian lattices and K3 surfaces; see [3] and its references. $N_{W_\Pi}(W_J)$ falls into 3 pieces: W_J itself, another Coxeter group W_Ω, and a group Γ_Ω of diagram automorphisms of W_Ω. The last two groups are called the "reflection" and "non-reflection" parts of the normalizer. Borcherds bounded the virtual cohomological dimension of Γ_Ω by $|J|$. Our theorems 1, 3 and 4 give stronger bounds, in terms of the types of the components of J rather than the number of nodes. There are choices involved in the definition of W_Ω and Γ_Ω, and our bound in theorem 3 applies regardless of how these choices are made (theorem 1 is a special case). Theorem 4 improves this bound when W_Ω is "maximal". In this case, when $J = D_5$ or $A_{m\,\mathrm{odd}}$, Γ_Ω turns out to either be free or have an index 2 subgroup that is free. This extends Brink's result [4] that Γ_Ω is free when $J = A_1$.

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We follow the notation of [3], and refer to [6] for general information about Coxeter groups. Suppose (W_Π, Π) is a Coxeter system, which is

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to say that W_Π is a Coxeter group and Π is a standard set of generators. The Coxeter diagram is the graph whose nodes are Π, with an edge between $s_i, s_j \in \Pi$ labeled by the order m_{ij} of $s_i s_j$, when $m_{ij} > 2$. W_Π acts isometrically on a real inner product space V_Π with basis (the simple roots) Π and inner products defined in terms of the m_{ij}. The (open) Tits cone K is an open convex subset of V_Π^* on which W_Π acts properly discontinuously with fundamental chamber C_Π. (Our C_Π and K are "missing" the faces corresponding to infinite parabolic subgroups of W_Π.) The standard generators act on V_Π^* by reflections across the hyperplanes containing the facets of C_Π, and they also act on V_Π by reflections. For a root α (i.e., a W_Π-image of a simple root) we write $\alpha^\perp$ for α's mirror, meaning the fixed-point set in K of the reflection associated to α.

Now let $J \subseteq \Pi$ be a spherical subdiagram, i.e., one corresponding to a finite subgroup of W_Π, and let $W_{\min}$ be the group generated by the reflections in W_Π that act trivially on $V_J \subseteq V_\Pi$. This is the "reflection" part of $N_{W_\Pi}(W_J)$, or rather the strictest possible interpretation of this idea. It corresponds to Borcherds' W_Ω in the case that the groups he calls Γ_Π and Γ_J are trivial; see the discussion after lemma 2. Let $J^\perp := \cap_{\alpha \in J} \alpha^\perp$, pick a component $C^\circ_{\min}$ of the complement of $W_{\min}$'s mirrors in $J^\perp$, and define $C_{\min}$ as its closure (in $J^\perp$). By definition, $W_{\min}$ is a Coxeter group, and the general theory of these groups shows that $C_{\min}$ is a chamber for it. The "non-reflection" part of $N_{W_\Pi}(W_J)$ means the subgroup $\Gamma_{\min}$ of W_Π preserving J (regarded as a set of roots) and sending $C_{\min}$ to itself. The reason for the first condition is to discard the trivial part of $N_{W_\Pi}(W_J)$, namely W_J itself. That is, $W_{\min}{:}\Gamma_{\min}$ is a complement to W_J in $N_{W_\Pi}(W_J)$. We write $\Gamma^\vee_{\min}$ for the subgroup of $\Gamma_{\min}$ acting trivially on J (equivalently, on V_J). The reason for passing to this (finite-index) subgroup is that $\Gamma_{\min}$ often contains torsion and therefore has infinite cohomological dimension for boring reasons.

Theorem 1. $\Gamma^\vee_{\min}$ *acts freely on a contractible cell complex of dimension at most*

$$\#A_1 + \#D_{m>4} + \#E_6 + \#I_2(5) + 2\big(\#A_{m>1} + \#D_4\big) \tag{1}$$

where $\#X_m$ means the number of components of J isomorphic to a given Coxeter diagram X_m. In particular, $\Gamma^\vee_{\min}$'s cohomological dimension is at most (1).

Borcherds' result [3, thm. 4.1] has $|J|$ in place of (1), but treats a more general group Γ_Ω, of which $\Gamma_{\min}$ is a special case. The more general case follows from this one, in theorem 3 below.

Proof. First we prove for $x \in C^{\circ}_{\min}$ that its stabilizer $\Gamma^{\vee}_{\min,x}$ is trivial. The W_Π-stabilizer of x is some W_Π-conjugate W_x of a spherical parabolic subgroup of W_Π. So W_x acts on V_Π as a finite Coxeter group. It is well-known that any vector stabilizer in such an action is generated by reflections, so the subgroup $W_{x,J}$ fixing J pointwise is generated by reflections. Observe that any reflection in $W_{x,J}$ lies in $W_{\min}$. Since x lies in the interior $C^{\circ}_{\min}$ of $C_{\min}$, it is fixed by no reflection in $W_{\min}$, so there can be no reflection in $W_{x,J}$, so $W_{x,J} = 1$. It is easy to see that $W_{x,J}$ contains $\Gamma^{\vee}_{\min,x}$, so we have proven that $\Gamma^{\vee}_{\min}$ acts freely on $C^{\circ}_{\min}$.

$C^{\circ}_{\min}$ is contractible because it is convex, and it obviously admits an equivariant deformation-retraction to its dual complex. So it suffices to show that the dual complex has dimension at most (1). Suppose $\phi \subseteq J^\perp$ is a face of a chamber of W_Π, with codimension in $J^\perp$ larger than (1); we must show $\phi \cap C^{\circ}_{\min} = \emptyset$. For some $w \in W_\Pi$, $w\phi$ is a face of C_Π whose corresponding set of simple roots $I' \subseteq \Pi$ contains $J' := w(J) \cong J$. By the codimension hypothesis on ϕ, $|I'| - |J'|$ is more than (1). Applying the lemma below to J' and I', we see that $W_{I'}$ contains a reflection r fixing J' pointwise. Since $r \in W_{I'}$, its mirror contains $w\phi$. So $w^{-1}rw$ is a reflection fixing J pointwise (so it lies in $W_{\min}$), whose mirror contains ϕ. Since $C^{\circ}_{\min}$ is a component of the complement of the mirrors of $W_{\min}$, it is disjoint from ϕ, as desired. $\square$

Lemma 2. *If J lies in a spherical Coxeter diagram $I \subseteq \Pi$, whose cardinality exceeds that of J by more than* (1)*, then W_I contains a reflection fixing J pointwise.*

Remark. Equality in (1) holds when I extends the A_m, D_m, E_6 and $I_2(5)$ components of J by $A_1 \to A_2$, $A_{m>1} \to D_{m+2}$, $D_4 \to E_6$, $D_{m>4} \to D_{m+1}$, $E_6 \to E_7$ and $I_2(5) \to H_3$. One can check in this case that the conclusion of the lemma fails.

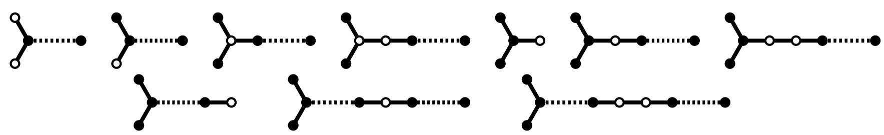
Proof. We may suppose $I = \Pi$, by discarding the rest of Π. Working one component at a time, it suffices to prove the lemma under the additional hypothesis that Π is connected. We now consider the various possibilities for Π, and suppose W_Π contains no reflections fixing V_J pointwise. That is, we assume $W_{\min} = 1$. In each case we will show that $|\Pi| - |J|$ is at most (1).

The $\Pi = A_n$ case is a model for the rest. Suppose the component of J nearest one end of Π has type A_m and does not contain that end. Then it must be adjacent to that end (since $W_{\min} = 1$), so together with the end it forms an A_{m+1}. We conjugate by the long word in $W(A_{m+1})$, which exchanges the two A_m diagrams in A_{m+1} and fixes the roots in the other components of J. The result is that we may suppose without loss that J contains that end of Π. Repeating the argument to move

the other components of J toward that end, we may suppose that there is exactly one node of Π between any two consecutive components of J. And the other end of Π is either in J or adjacent to it. It is now clear that $|\Pi| - |J|$ is the number of components of J, or one less than this. Since every component of J has type A, $|\Pi| - |J|$ is at most (1). This finishes the proof in the $\Pi = A_n$ case.

If $\Pi = B_n = C_n$ then we begin by shifting any type A components of J as far as possible from the double bond. If J has no B_m then J contains one end of the double bond, and we get $|\Pi| - |J|$ equal to the number of components of J, all of which have type A. If J has a B_m then the node after it (if there is one) must be adjacent to some type A component of J. This is because $W(B_{m+1})$ contains a reflection acting trivially on V_{B_m}. This is easy to see in the model of $W(B_{m+1})$ as the isometry group of $\mathbb{Z}^{m+1}$. It follows that $|\Pi| - |J|$ is the number of components of J of type A.

In the $\Pi = D_{n>3}$ case, one can use the shifting trick to reduce to one of the cases

(2) 

where the filled nodes are those in J and the dashes indicate a chain of nodes with no two consecutive unfilled nodes. (Except for the dashes on the left in the last 3 diagrams, which indicate chains of filled nodes.) In every case we get

$$|\Pi| - |J| \leq \#A_1 + \#D_{m\geq 4} + 2\,\#A_{m>1}.$$

The most interesting case is $A_{n-2} \to D_n$, at the top left.

We will treat the case $\Pi = E_8$ and leave the similar E_6 and E_7 cases to the reader. If J has a D_4, D_5 or E_6 component, then it must also have a type A component, and then $|\Pi| - |J| \leq 2\,\#D_4 + \#D_5 + \#A_{m\geq 1}$, as desired. J cannot be D_6 or E_7, because then $W_{\min}$ would contain the reflection in the lowest root of E_8, which extends E_8 to the affine diagram $\tilde{E}_8$. So we may suppose J's components have type A. In order for $|\Pi| - |J|$ to exceed (1), we must have $J = A_{m\leq 5}$, A_3A_1, A_2A_1 or $A_1^{m\leq 3}$. But none of these cases can occur, because in each of them we may shift J's components around so that some node of Π is not joined to J.

The remaining cases are $\Pi = F_4$, H_3, H_4 and I_2, the last case including $G_2 = I_2(6)$. The facts required to treat these cases are that if $J = B_2$ or B_3 in $\Pi = F_4$ then $W_{\min}$ contains a reflection, and similarly in the $J = H_3 \subseteq H_4 = \Pi$ case. The first fact is visible inside a B_3 or

B_4 root system inside F_4. To see the second, observe that the root stabilizer in H_4 contains Coxeter groups of types A_2 and $I_2(5)$, visible in the centralizers of the two end reflections of H_4 (which are conjugate). So the root stabilizer can only be $W(H_3)$, which is to say that the H_3 root system is orthogonal to a root. $\square$

The greater generality obtained by Borcherds is the following. Let Γ_Π be a group of diagram automorphisms of Π, acting on V_Π and K in the obvious way. The goal is to understand $N_{W_\Pi:\Gamma_\Pi}(W_J)$. Again we discard the boring part of this normalizer by passing to the subgroup W'_J preserving the set of roots $J \subseteq \Pi$. Let W_Ω be any subgroup of W'_J which contains $W_{\min}$ and is generated by elements which act on $J^\perp$ by reflections. We define C°_Ω, C_Ω and Γ_Ω as for $C^\circ_{\min}$, $C_{\min}$ and $\Gamma_{\min}$, and define $\Gamma^\vee_\Omega$ as the subgroup of $\Gamma_\Omega \cap W_\Pi$ acting trivially on J. (Borcherds left $\Gamma^\vee_\Omega$ unnamed and defined W_Ω in terms of auxiliary groups $R \trianglelefteq \Gamma_J \subseteq \operatorname{Aut} J$; his W_Ω has the properties assumed here.) The inclusion $W_{\min} \subseteq W_\Omega$ is the source of the subscript "min", but note that $C_{\min}$ and $\Gamma_{\min}$ are *larger* than C_Ω and Γ_Ω. We can now recover Borcherds' result [3, thm. 4.1] with our (1) in place of $|J|$.

Theorem 3. *Theorem 1 holds with* $\Gamma^\vee_{\min}$ *replaced by* $\Gamma^\vee_\Omega$.

Proof. The freeness of the action follows from the same argument. (This is why $\Gamma^\vee_\Omega$ is defined as a subgroup of $\Gamma_\Omega \cap W_\Pi$ rather than just Γ_Ω.) The essential point for the rest of the proof is that W_Ω contains $W_{\min}$, so the decomposition of $J^\perp$ into chambers of W_Ω refines that of $W_{\min}$. This shows $C^\circ_\Omega \subseteq C^\circ_{\min}$. So the dual complex of C°_Ω has dimension at most that of $C^\circ_{\min}$, and we can apply theorem 1. $\square$

The point of considering W_Ω rather than $W_{\min}$ is that it is larger and so Γ_Ω will be smaller than $\Gamma_{\min}$. This is good since the nonreflection part is more mysterious than the reflection part. So it is natural to define $W_{\max}$ by setting $\Gamma_\Pi = 1$ and taking W_Ω as large as possible, i.e., $W_{\max}$ is the subgroup of W'_J generated by the transformations which act on $J^\perp$ by reflections.

This is the largest possible "universal" W_Ω, although a larger W_Ω is possible if Π admits suitable diagram automorphisms. For example, Γ_Π might contain elements acting on C_Π by reflections. I don't know other examples, although probably there are some.

We define $C^\circ_{\max}$, $C_{\max}$, $\Gamma_{\max}$ and $\Gamma^\vee_{\max}$ as above. The next theorem follows from lemma 5 in exactly the same way that theorem 1 follows from lemma 2.

Theorem 4. *The dimension of the dual complex of $C^\circ_{\max}$, hence the cohomological dimension of $\Gamma^\vee_{\max}$, is bounded above by*

$$\#D_5 + \#A_{m\,\mathrm{odd}} + 2\,\#A_{m\,\mathrm{even}}. \tag{3}$$

□

Remarks. **(i)** If J has no A_m or D_5 component then $\Gamma^\vee_{\max} = 1$ and $\Gamma_{\max}$ is finite. This is Borcherds' [3, example 5.6]. **(ii)** If $J = D_5$ or $A_{m\,\mathrm{odd}}$ then $\Gamma^\vee_{\max} \subseteq N_{W_\Pi}(W_J)$ is free. Also, since $|\operatorname{Aut} J| \leq 2$, $\Gamma^\vee_{\max}$ has index 1 or 2 in $\Gamma_{\max}$. Therefore the non-reflection part $\Gamma_{\max}$ of $N_{W_\Pi}(W_J)$ has a free subgroup of index 1 or 2. **(iii)** If $J = A_1$ then $\Gamma_{\min} = \Gamma^\vee_{\min} = \Gamma_{\max} = \Gamma^\vee_{\max}$ has cohomological dimension ≤ 1. This recovers Brink's result [4] that $\Gamma_{\min}$ is free. **(iv)** If $J = A_{m\,\mathrm{even}}$ then the conclusion dim(dual of $C^\circ_{\min}$) ≤ 2 suggests that $\Gamma_{\max}$ is often comprehensible, like the $J = A_6$ example of [3, example 5.4].

Lemma 5. *If J lies in a spherical Coxeter diagram $I \subseteq \Pi$, whose cardinality exceeds that of J by more than* (3)*, then W_I contains an element preserving the set J of roots and acting on $J^\perp$ by a reflection.*

Proof. This is essentially the same as for lemma 2, using the following additional ingredients. For example, when $I = D_n$ one can use them to show that the 5th, 7th, 8th and 10th cases of (2) are impossible, while the first can only occur when n is even.

First, if $J = E_6 \subseteq E_7 = I$ then W_I contains the negation of V_I, which we follow by the long word in W_J to send $-J$ back to J. The composition is the claimed element of W_I. The same argument applies if $J = I_2(5) \subseteq H_3 = I$.

Second, if $J = A_{m\,\mathrm{odd}} \subseteq D_{m+2} = I$ as in the first diagram of (2), then consider the long word in W_I. It negates J and exchanges and negates the two simple roots in $I - J$. Following this by the long word in W_J yields the claimed element of W_I. (When m is even, the long word in W_I negates the simple roots in $I - J$ without exchanging them, so it doesn't act on $J^\perp$ by a reflection.)

Third, if $J = D_{m\geq 3} \subseteq D_{m+1} = I$ then consider the model of W_I as the group generated by permutations and evenly many negations of $m+1$ coordinates, with W_J the corresponding subgroup for the first m coordinates. Letting σ be the negation of the last two coordinates, and following it by the element of W_J sending $\sigma(J)$ back to J, gives the claimed element of W_I. □

There is a nice geometrical interpretation of the freeness of $\Gamma_{\min}$ in the case $J = A_1$, developed further in [1]. Namely, the natural map $C^\circ_{\min} \to C^\circ_{\min}/\Gamma_{\min} \subseteq K/W_\Pi = C_\Pi$ is the universal cover of its image.

The image is got by discarding all the codimension 2 faces of C_Π corresponding to even bonds in Π, discarding all codimension 3 faces, and taking the component corresponding to J. This identifies $\Gamma_{\min}$ with the fundamental group of J's component of the "odd" subgraph of Π in a natural manner.

One can extend this picture to the case $J \neq A_1$, but complications arise. First, one must take W_Ω to be normal in $W_\Pi{:}\Gamma_\Pi$. Second, while $C_\Omega^\circ \to C_\Omega^\circ/\Gamma_\Omega^\vee$ is a covering space, the image $C_\Omega^\circ/\Gamma_\Omega$ of C_Ω° in C_Π is the quotient of $C_\Omega^\circ/\Gamma_\Omega^\vee$ by the finite group $\Gamma_\Omega/\Gamma_\Omega^\vee$. Usually, $C_\Omega^\circ \to C_\Omega^\circ/\Gamma_\Omega$ is only an orbifold cover since Γ_Ω often has torsion. The top-dimensional strata of $C_\Omega^\circ/\Gamma_\Omega^\vee$ correspond to the "associates" of the inclusion $J \to \Pi$ in the sense of [3] and [5]. Suppose $J' \subseteq \Pi$ is (the image of) an associate and I' is a spherical diagram containing it. Then the face of C_Π corresponding to I', minus lower-dimensional faces, lies in $C_\Omega^\circ/\Gamma_\Omega$ just if $W_{I'}$ contains no element preserving J', acting on it in a manner constrained by the choice of W_Ω, and acting on $J'^\perp$ by a reflection. From this perspective, lemmas 2 and 5 amount to working out two cases of Borcherds' notion of "R-reflectivity". The orbifold structure on $C_\Omega^\circ/\Gamma_\Omega$ is essentially the same information as Borcherds' classifying category for Γ_Ω.

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